Communication Theory II

Lecture 4: Review on Fourier analysis and probabilty theory

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Course Website

- **http://lms.mans.edu.eg/eng/**
- The site contains the lectures, quizzes, homework, and open forums for feedback and questions
- Log in using your name and password
- Password for quizzes: third
- One page quiz: for less download time

Lecture Outlines

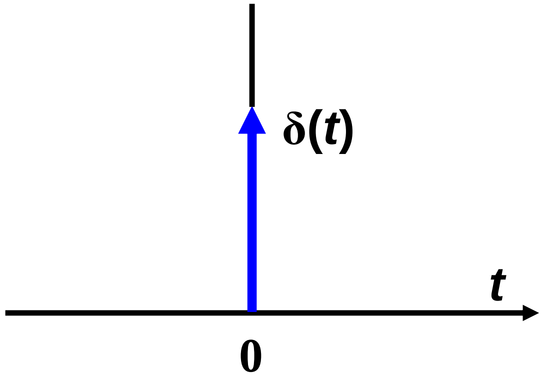
- Review on Fourier analysis of signals and systems
  - The Dirac delta function
  - Fourier transform of periodic signals
  - Transmission of signals through LTI systems
  - Hilbert transform
- Review on probability theory
  - Deterministic vs. probabilistic mathematical models
  - Probability theory, random variables, and the distribution functions
  - The concept of expectation and second order statistics
  - Characteristic function, the center limit theory and the Bayesian interface

The Dirac Delta Function (Unit Impulse)

Dirac delta function:

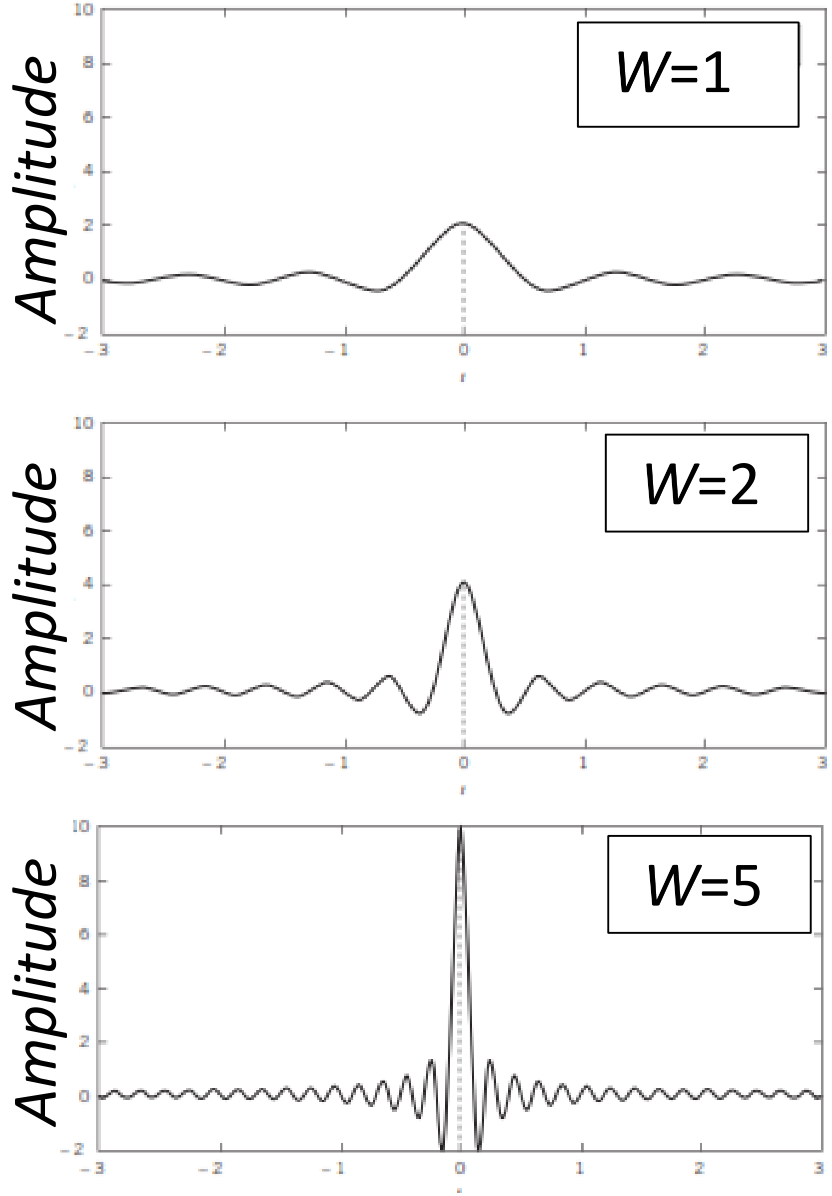
$\delta(t) = 0$ for $t \neq 0$ and

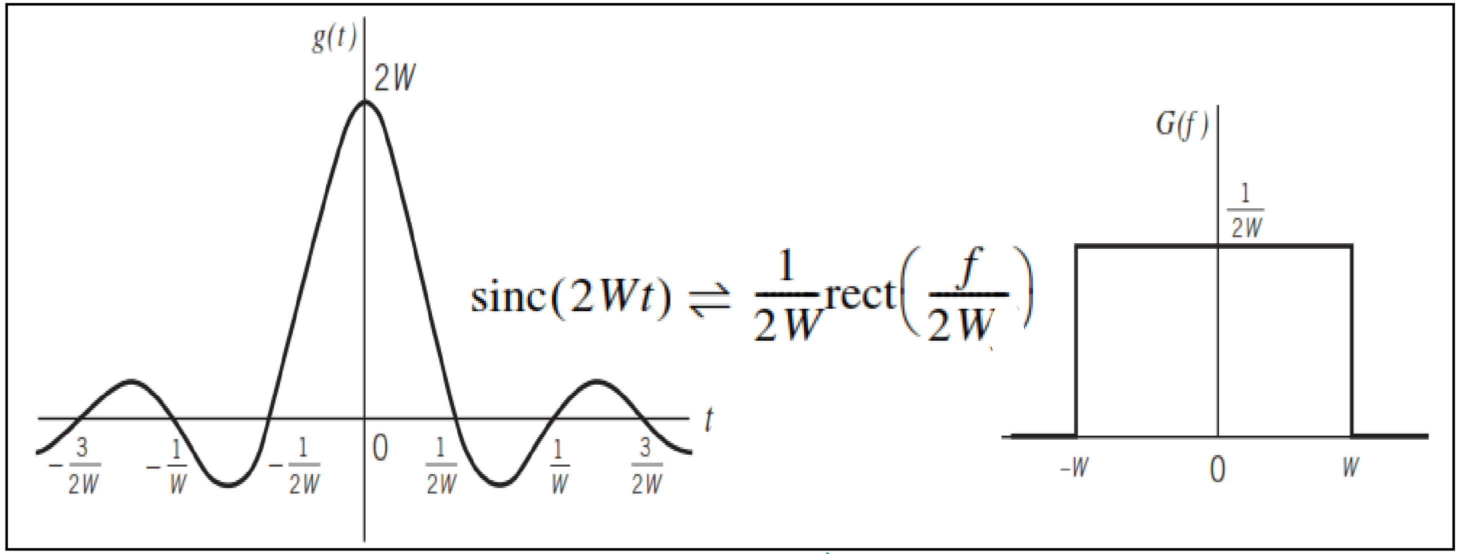
$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1$$



- An even function of time t, centered at the origin $t = 0$
- ***Sifting property:*** sifts out the value $g(t0)$ of the function $g(t)$ at time $t = t0$, where $\int_{-\infty}^{\infty} g(t)\,\delta(t - t_0)\,\mathrm{d}t = g(t_0)$
- ***Replication property:*** convolution of any function with the delta function leaves that function unchanged $\int_{-\infty}^{\infty} g(\tau)\,\delta(t - \tau)\,\mathrm{d}\tau = g(t)$

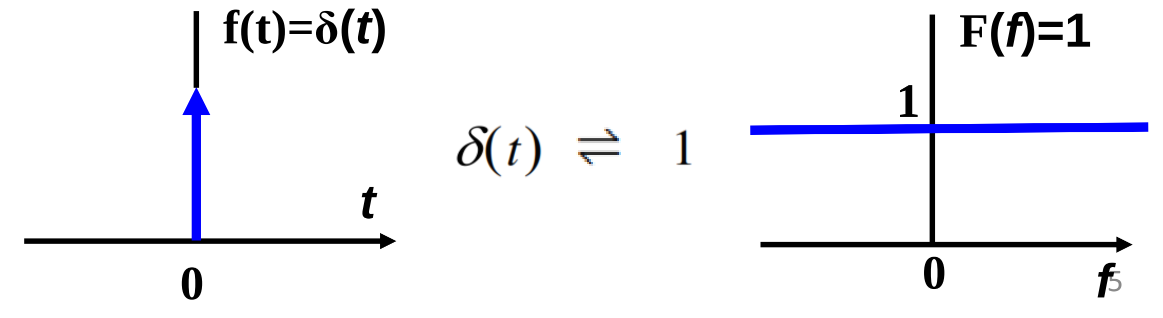
The Dirac Delta Function (cont'd)





$$\mathrm{sinc}(2Wt) \rightleftharpoons \frac{1}{2W}\mathrm{rect}\left(\frac{f}{2W}\right)$$

$$\delta(t) = \lim_{W \to \infty} 2W\,\mathrm{sinc}(2Wt)$$



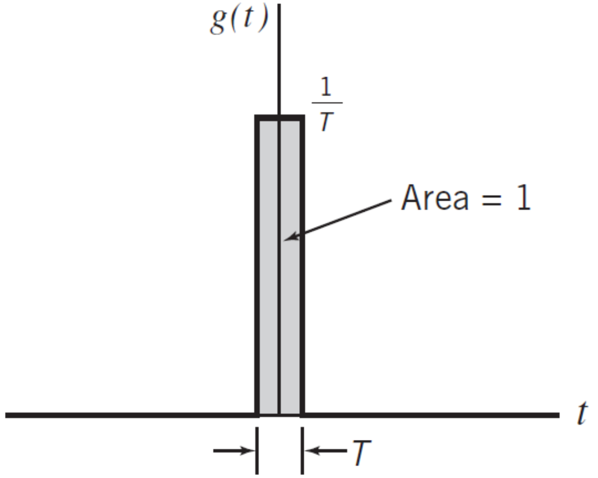
$$\delta(t) \rightleftharpoons 1$$

The Dirac Delta Function (cont'd)

The delta function may be viewed as the limiting form of a pulse of unit area (symmetric with respect to the origin) as the duration of the pulse approaches zero

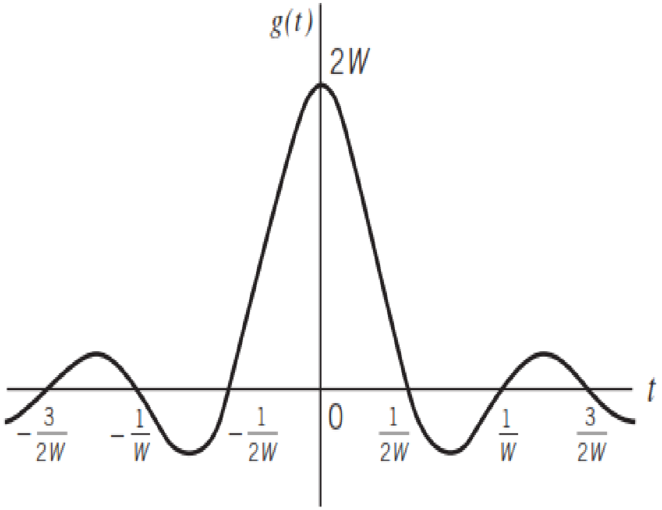
$$\delta(t) = \lim_{T \to 0} \frac{1}{T} \operatorname{rect}\left(\frac{t}{T}\right)$$

Rectangular impulse



$$\delta(t) = \lim_{W \to \infty} 2W \operatorname{sinc}(2Wt)$$

Sinc impulse



T=1/2W

Existence of Fourier Transform

For the Fourier transform of a signal $g(t)$ to exist, it is sufficient but not necessary that the nonperiodic signal $g(t)$ satisfies three *Dirichlet's conditions* of its own:

1. The function $g(t)$ is single valued, with a finite number of maxima and minima in any finite time interval.
2. The function $g(t)$ has a finite number of discontinuities in any finite time interval.
3. The function $g(t)$ is absolutely integrable; that is,

$$\int_{-\infty}^{\infty} |g(t)| \mathrm{d}t < \infty$$

- Physical realizability is a sufficient condition for the existence of a Fourier transform (e.g., all energy signals are Fourier transformable).

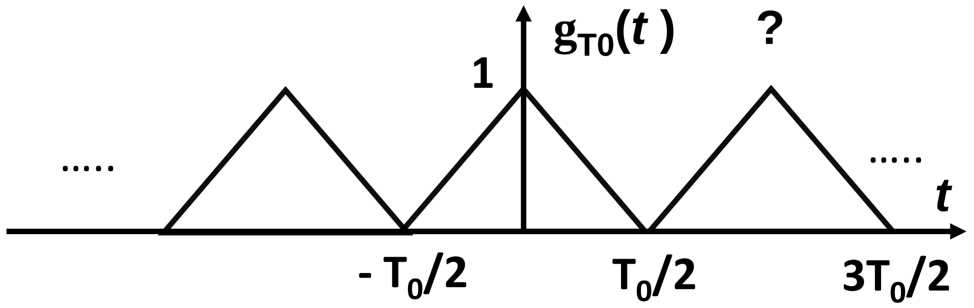
Condition for energy signal: $\int_{-\infty}^{\infty} |g(t)|^2 \mathrm{d}t < \infty$

- What about power and periodic signals? Do they have a Fourier transform?

Fourier Transform of Periodic Signals

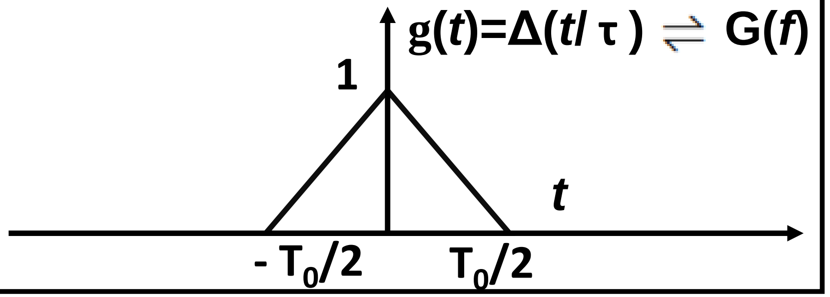
- Can Fourier transform works for periodic or power signals?

Periodic signal (power)



$$g(t) = \begin{cases} g_{T_0}(t), & -\frac{T_0}{2} < t \leq \frac{T_0}{2} \\ 0, & \text{elsewhere} \end{cases}$$

Aperiodic signal with computable G(*f*)

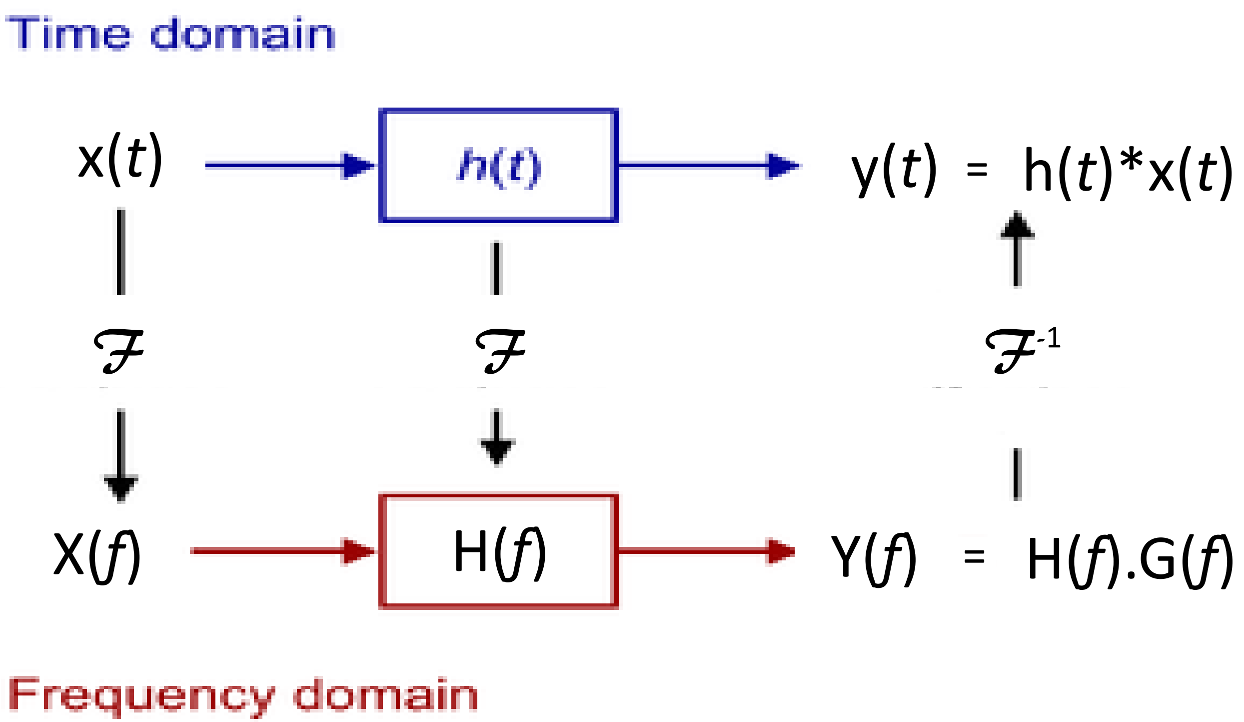


Time shift property

$$g_{T_0}(t) = \sum_{m=-\infty}^{\infty} g(t - mT_0) \longrightarrow G_{T_0}(f) = G(f) \sum_{m=-\infty} \exp(-j2\pi m f T_0), \qquad -\infty < f < \infty$$

Fourier Transform for Analyzing Signals and Systems

- Provides the mathematical link between the time domain of a signal (waveform) and its frequency domain (spectrum)
- Time and frequency response of a linear time-invariant (LTI) system defined in terms of its impulse response and frequency response, respectively



LTI system

If $x_1(n) \rightarrow \boxed{h(n)} \rightarrow y_1(n)$

and $x_2(n) \rightarrow \boxed{h(n)} \rightarrow y_2(n)$

Then $a_1y_1(n)+a_2y_2(n)=H\{a_1x_1(n)+a_2x_2(n)\}$

Where a1, a2 are scalars

(a)

If $x_1(n) \rightarrow \boxed{h(n)} \rightarrow y_1(n)$

Then $x_1(n-k) \rightarrow \boxed{h(n)} \rightarrow y_1(n-k)$

(b)

A (a) linear and (b) a time invariant system

Hilbert Transform

- **A quadrature filter:** phase angles of all components of a given signal are shifted by ±90°

Hilbert transform pair

$\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(\tau)}{t-\tau}\, d\tau$	$g(t) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\hat{g}(\tau)}{t-\tau}\, d\tau$

- Example: $g(t) = \boldsymbol{\delta(t)}$

$$\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(\tau)}{t-\tau}\, d\tau = \frac{1}{\pi t}$$

Using replication property of the delta function

Table 2.3 Hilbert-transform pairs*

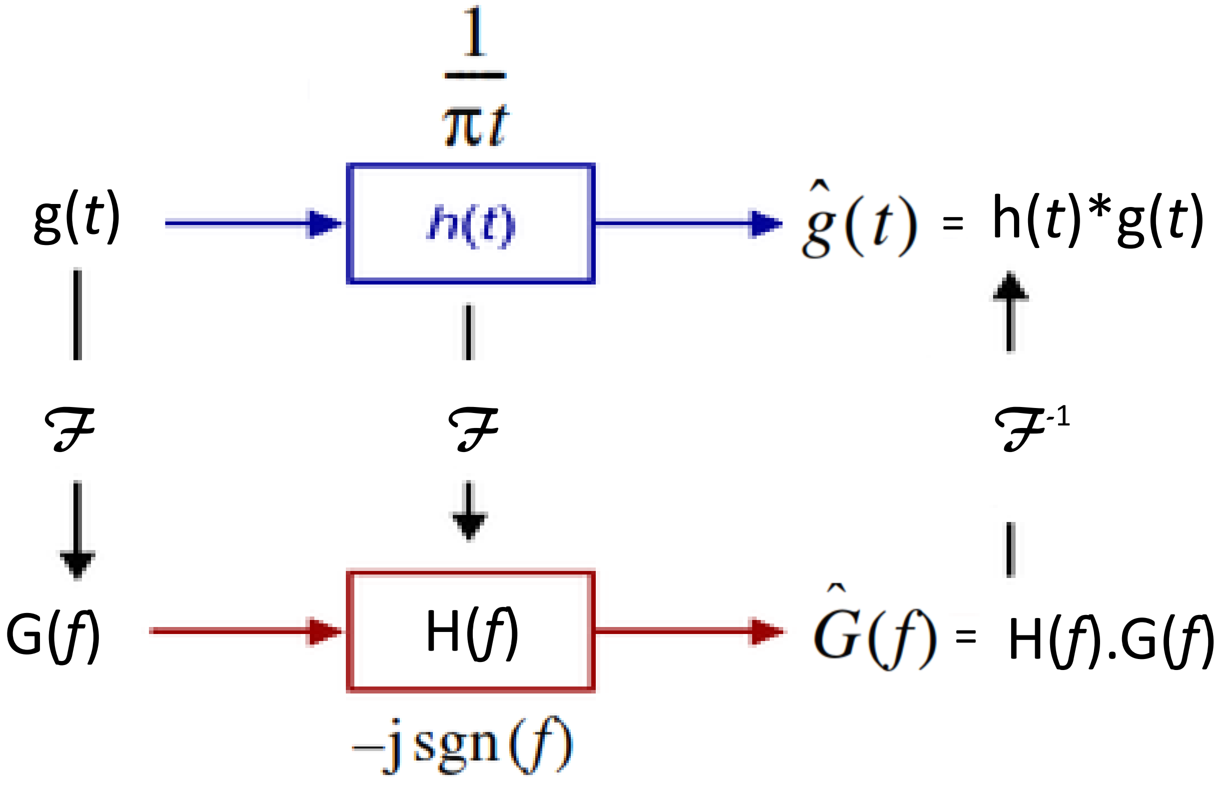
Time function	Hilbert transform	
1. $m(t)\cos(2\pi f_c t)$	$m(t)\sin(2\pi f_c t)$	$m(t)$ is band limited to the interval $-W \le f \le W$, where $W < f_c$.
2. $m(t)\sin(2\pi f_c t)$	$-m(t)\cos(2\pi f_c t)$	$m(t)$ is band limited to the interval $-W \le f \le W$, where $W < f_c$.
3. $\cos(2\pi f_c t)$	$\sin(2\pi f_c t)$	
4. $\sin(2\pi f_c t)$	$-\cos(2\pi f_c t)$	
5. $\dfrac{\sin t}{t}$	$\dfrac{1-\cos t}{t}$	
6. $\mathrm{rect}(t)$	$-\dfrac{1}{\pi}\ln\left\lvert\dfrac{t-1/2}{t+1/2}\right\rvert$	
7. $\delta(t)$	$\dfrac{1}{\pi t}$	
8. $\dfrac{1}{1+t^2}$	$\dfrac{t}{1+t^2}$	
9. $\dfrac{1}{t}$	$-\pi\delta(t)$	

Notes: $\delta(t)$ denotes Dirac delta function; $\mathrm{rect}(t)$ denotes rectangular function; ln denotes natural logarithm.

* In the first two pairs, it is assumed that $m(t)$ is band limited to the interval $-W \le f \le W$, where $W < f_c$.

Hilbert transform (cont'd)

- The function $\hat{g}(t)$ may be interpreted as $g(t) * \frac{1}{\pi t} \rightleftharpoons G(f) \cdot -j\,\mathrm{sgn}(f)$



$$\frac{1}{\pi t} \rightleftharpoons -j\,\mathrm{sgn}(f)$$

$$\mathrm{sgn}(f) = \begin{cases} 1, & f > 0 \\ 0, & f = 0 \\ -1, & f < 0 \end{cases}$$

Hilbert Transform properties

1. $$|G(f)| = |\hat{G}(f)|$$

2. $$\arg[G(f)] = -\arg\{\hat{G}(f)\}$$

3. A Hilbert pair is orthogonal over the entire time interval

$$\int_{-\infty}^{\infty} g(t)\hat{g}(t)\mathrm{d}t = 0$$

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*Note that soft materials are from COMPE 467 - Pattern Recognition, www.atilim.edu.tr/~mkahraman/2_Foundations.ppt

Deterministic vs. Probabilistic Models

- **Deterministic mathematical model:** if there is no uncertainty about its time-dependent behavior at any instant of time
- **Example:** linear time-invariant systems
- **Problem:** underlying physical phenomenon involves too many unknown factors (e.g., noise, fading, interference)
- **Probabilistic model** accounts uncertainty in mathematical terms
- Probabilistic models are intended to assign probabilities to the collections (sets) of possible outcomes of random experiments

Probability Theory: Sets

- Probability makes extensive use of set operations
- A **set** is a collection of objects, which are the **elements** of the set
- If S is a set and x is an element of S, we write $x \in S$, otherwise $x \notin S$
- A set can have no elements, in which case it is called the **empty set**, denoted by $\emptyset$
- The set of all possible outcomes of an experiment is the **sample space** or **universe**, denoted $\boldsymbol{\Omega}$
- An **event A** is a (set of) possible outcomes of the experiment, and corresponds to a subset of $\boldsymbol{\Omega}$

Probability Theory: Sets (cont'd)

- Sets can be specified as

$$S = \{x_1, x_2, \ldots, x_n\}.$$

- For example,
  - The set of possible outcomes of a die roll is {1, 2, 3, 4, 5, 6},
  - The set of possible outcomes of a coin toss is {H, T}, where H stands for "heads" and T stands for "tails"

Probability Theory: Sets (cont'd)

- Alternatively, we can consider the set of all *x* that have a certain property *P*, and denote it by

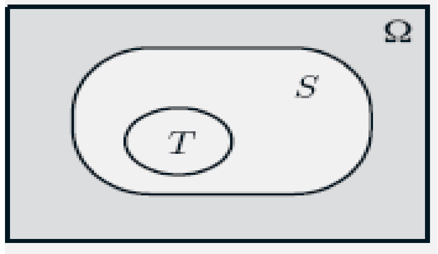
$$\{x \mid x \text{ satisfies } P\}.$$

(The symbol "|" is to be read as "such that.")

- For example,
  - the set of even integers can be written as $\{k \mid k/2$ is integer$\}$.

Probability Theory: Sets Operations

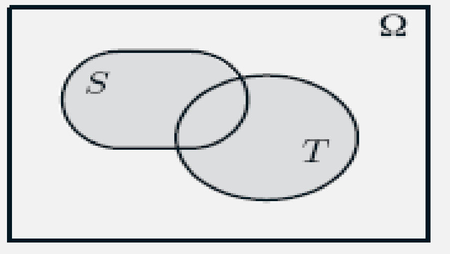
- **Complement:**
  - The **complement** of a set S, with respect to the universe Ω, is the set $\{x \in \Omega \,|\, x \notin S\}$ of all elements of Ω that do not belong to S, and is denoted by S^c.



T is a subset of S
$T \subset S$

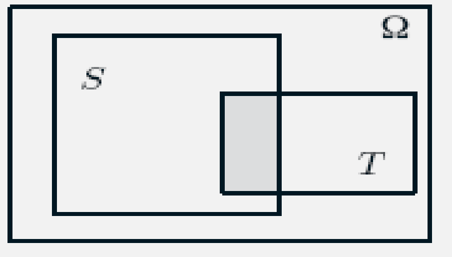
- **Union:**

$$S \cup T = \{x \,|\, x \in S \text{ or } x \in T\},$$



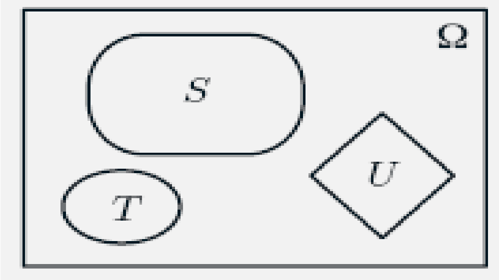
- **Intersection:**

$$S \cap T = \{x \,|\, x \in S \text{ and } x \in T\}.$$



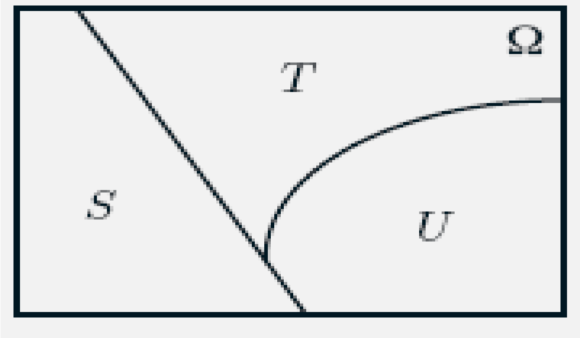
Probability Theory: Sets Operations

- **Disjoint sets:**
  - two sets are said to be **disjoint** if their intersection is empty



- **Partition of a set *A*:**
  - A collection of disjoint subsets of *A*, where their union is *A*

Partition of Ω



Probability measure

- A probability law / measure is a function p(*A*) that assigns a nonnegative value to event *A*
  - based on the expected proportion of number of times that event *A* is actually likely to happen: encodes our belief in the likelihood event A occurring when the experiment is conducted

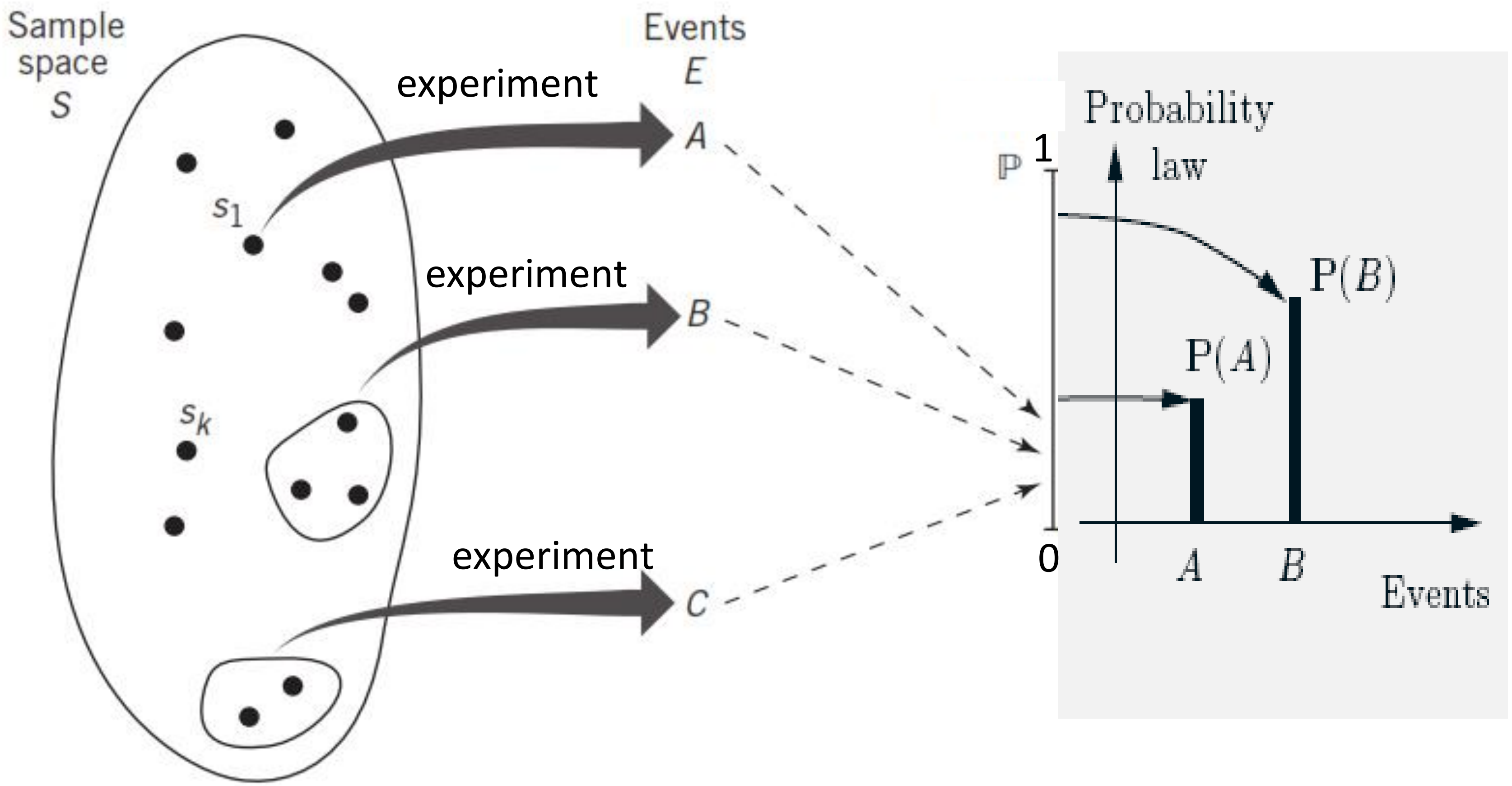


Figure 3.4 Illustration of the relationship between sample space, events, and probability

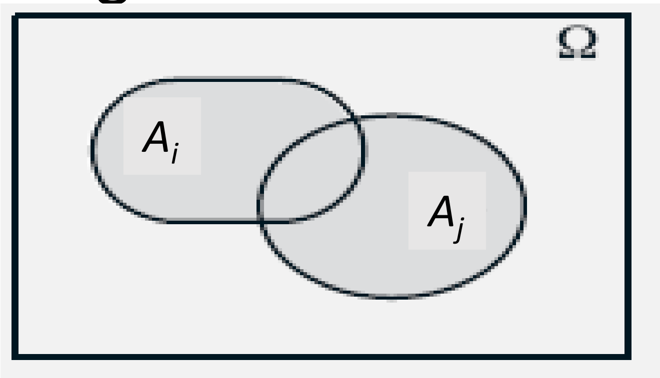
Probability Theory: Axioms of Probability

Probability Axioms

1. **(Nonnegativity)** $\mathbf{P}(A) \geq 0$, for every event A.
2. **(Additivity)** If A and B are two disjoint events, then the probability of their union satisfies
$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B).$$
More generally, if the sample space has an infinite number of elements and $A_1, A_2, \ldots$ is a sequence of disjoint events, then the probability of their union satisfies
$$\mathbf{P}(A_1 \cup A_2 \cup \cdots) = \mathbf{P}(A_1) + \mathbf{P}(A_2) + \cdots.$$
3. **(Normalization)** The probability of the entire sample space Ω is equal to 1, that is, $\mathbf{P}(\Omega) = 1$.

Probability Theory: Axioms of Probability (cont'd)

- The probability function **P(A)** must satisfy the following:



$$0 \le P(A_i) \le 1,$$

$$P(\Omega) = \sum P(A_i)$$

$$A_i \cap A_j = \phi \Rightarrow \quad P(A_i \cup A_j) = P(A_i) + P(A_j),$$

$$A_i \cap A_j \ne \phi \Rightarrow \quad P(A_i \cup A_j) = P(A_i) + P(A_j) - P(A_i \cap A_j),$$

Example 1

Example Consider an experiment involving a single coin toss. There are two possible outcomes, heads (H) and tails (T). The sample space is $\Omega = \{H, T\}$, and the events are

$$\{H, T\},\ \{H\},\ \{T\},\ \emptyset.$$

If the coin is fair, i.e., if we believe that heads and tails are "equally likely," we should assign equal probabilities to the two possible outcomes and specify that $\mathbf{P}(\{H\}) = \mathbf{P}(\{T\}) = 0.5$.

Example 1 (cont'd)

The additivity axiom implies that

$$\mathbf{P}\big(\{H,T\}\big) = \mathbf{P}\big(\{H\}\big) + \mathbf{P}\big(\{T\}\big) = 1,$$

which is consistent with the normalization axiom. Thus, the probability law is given by

$$\mathbf{P}\big(\{H,T\}\big) = 1, \qquad \mathbf{P}\big(\{H\}\big) = 0.5, \qquad \mathbf{P}\big(\{T\}\big) = 0.5, \qquad \mathbf{P}(\varnothing) = 0,$$

and satisfies all three axioms.

Example 2

Consider another experiment involving three coin tosses. The outcome will now be a 3-long string of heads or tails. The sample space is

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

We assume that each possible outcome has the same probability of 1/8.

Example 2 (cont'd)

Consider, as an example, the event

$$A = \{\text{exactly 2 heads occur}\} = \{HHT, HTH, THH\}.$$

Using additivity, the probability of A is the sum of the probabilities of its elements:

$$\begin{aligned}
\mathbf{P}\big(\{HHT, HTH, THH\}\big) &= \mathbf{P}\big(\{HHT\}\big) + \mathbf{P}\big(\{HTH\}\big) + \mathbf{P}\big(\{THH\}\big) \\
&= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\
&= \frac{3}{8}.
\end{aligned}$$

Random Variables

- Real experiments involve using one or more real valued quantities called random variables
- The outcomes are associated with some numerical values of interest (random variable).
- **Example:** if the experiment is the selection of students from a given population, we may wish to consider their grade point average.

The **random variable** is a function whose domain is a sample space and whose range is some set of real numbers

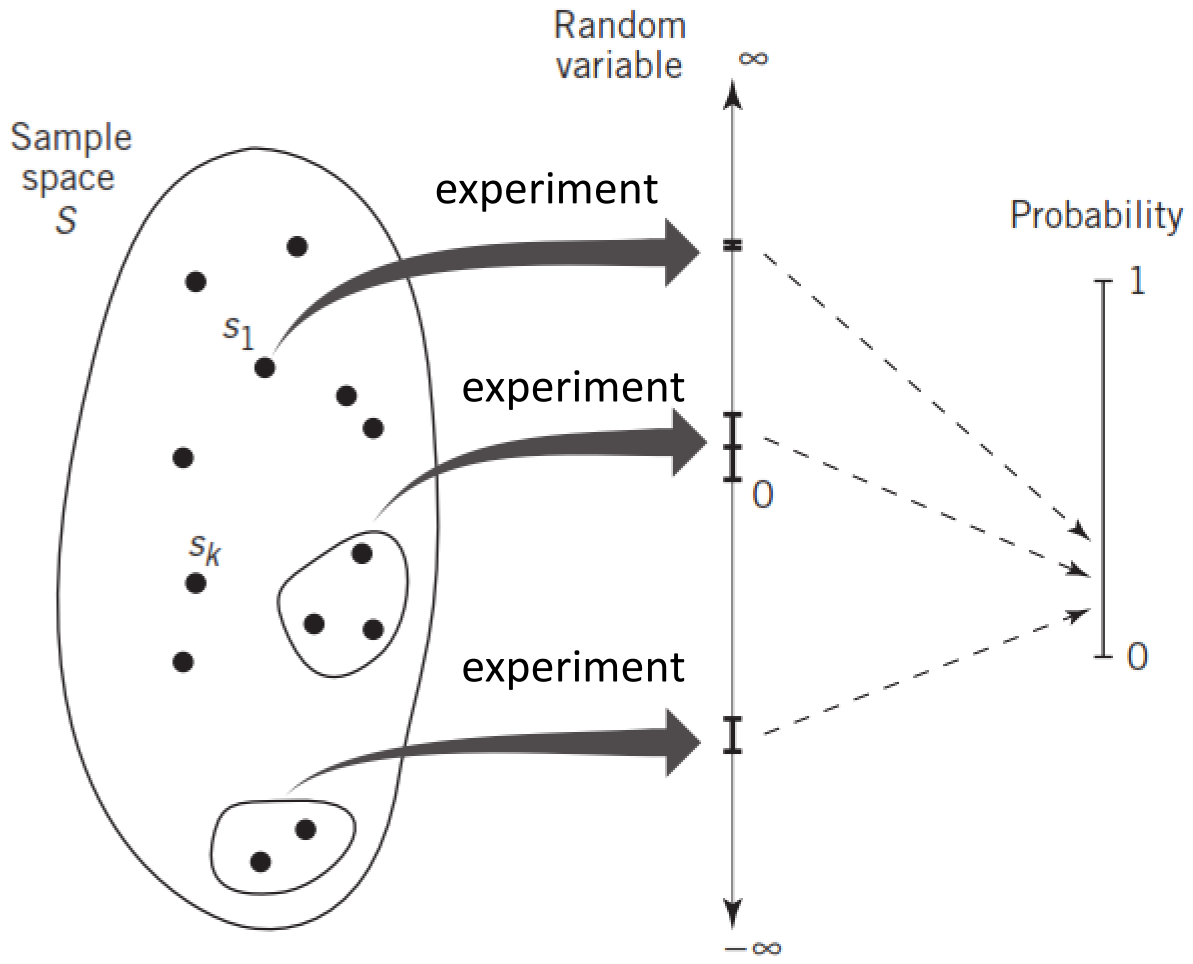


Figure 3.7 Illustration of the relationship between sample space, random variables, and probability.

Discrete and Continuous Random Variables

- Random variables can **discrete**, e.g., the number of heads in three consecutive coin tosses, or **continuous**, the weight of a class member.

Questions